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Properties of Fully Developed Chaos in One-Dimensional Maps

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We consider single-humped symmetric one-dimensional maps generating fully developed chaotic iterations specified by the property that on the attractor the mapping is everywhere two to one. To calculate the probability distribution function, and in turn the Lyapunov exponent and the correlation function, a perturbation expansion is developed for the invariant measure. Besides deriving some general results, we treat several examples in detail and compare our theoretical results with recent numerical ones. Furthermore, a necessary condition is deduced for the probability distribution function to be symmetric and an effective functional iteration method for the measure is introduced for numerical purposes.

KEY WORDS: Chaos; probability distribution; perturbation theory; Lyapunov exponent; correlation function.

1. INTRODUCTION AND SUMMARY

In recent years there has been considerable interest in nonlinear processes exhibiting chaotic behavior, especially in one-dimensional noninvertible maps. Such discrete models are related to several phenomena as turbulence, irregular behavior in electronic circuits, chemical reactions, optical systems, etc. In particular, single-humped maps, which exhibit a very rich variety of behavior when changing a control parameter, have often provided a suit-

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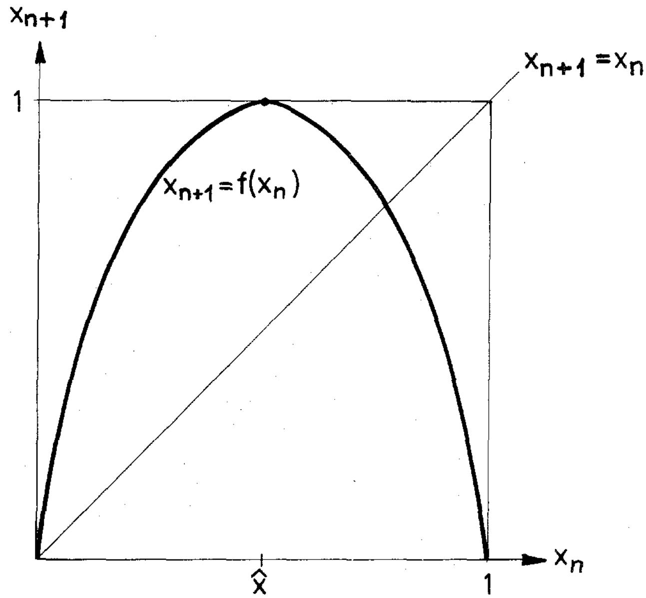

Fig. 1. Single-humped map leading to fully developed chaotic dynamics.

able framework for modeling real systems. (See Refs. 1–4, which contain extensive further references.)

We consider one-dimensional (1D) maps with a single extremal point $\hat{x}$ and the property $f(0) = f(f(\hat{x})) = 0$ [see Fig. 1 and for further specifications of $f(x)$ the first paragraph of Section 2]. The iteration produced by such maps is chaotic for almost all initial values on the interval which is mapped onto itself.[1,5–7] The purpose of the present paper is the examination of this chaotic state, which we call fully developed chaos. It can be characterized by the property that the attractor can be decomposed into two intervals, each of which is mapped to the whole attractor in one step. A fully developed chaotic attractor can be observed in the logistic map $f(x) = rx(1 - x)$ when the control parameter r equals four. More generally, it is the final stage of the evolution of the attractor (from fixed point through periodic orbits to chaotic attractors) of a 1D single-humped map. Fully developed chaos can also be found in parameter-controlled maps at the band merging points and also at the crisis points[1,5,8–11] if one considers a suitable iterate of the original map. Consequently, our investigation is relevant for a large class of maps.

Lyapunov characteristic exponent (LCE) and the correlation function are widely used quantities for characterizing iterations. It can be shown that if the probability distribution function belonging to a symmetric[3] map is also symmetric, which we call the doubly symmetric case, the process is delta-correlated[(5)] and the value of the LCE equals $\ln 2$. This is the case, e.g., for the logistic map mentioned above, while according to numerical experiments on other symmetric maps close to the logistic one exhibiting fully developed chaos, the LCE and the correlation function deviate slightly from those cited above.[(5,10–15,18)] It is appealing to suppose that this phenomenon is due to the nonsymmetric character of the probability distribution functions belonging to such maps.

In this paper we derive a necessary condition for a symmetric map to be a doubly symmetric one. Namely, if the map has a kth-order maximum its derivative at the origin should be equal to 2^k. It is then obvious that doubly symmetric maps represent exceptional cases and there should be many symmetric maps exhibiting fully developed chaos in the vicinity of a doubly symmetric one, which do not have symmetric probability distributions (and consequently are not conjugate[(5)] to doubly symmetric maps). To demonstrate this we introduce a functional iteration method which turns out to be very effective numerically and provides results of high accuracy.

Our main interest in this paper is to study the change of the behavior of symmetric maps exhibiting fully developed chaos when a parameter, measuring the deviation from a particular map with known properties, is continuously changed. For this purpose we develop a perturbative method based on the equation for the invariant measure. We derive some general results and we treat some examples in detail. The main results are as follows.

We consider only symmetric maps and suppose that both the unperturbed and the perturbed maps exhibit fully developed chaos. It is shown that a small perturbation of a doubly symmetric map $f_0(x)$ can be split in such a way that a part of it has the form $\epsilon\mathscr{G}(f_0(x))$, $\mathscr{G}(z)=\mathscr{G}(1-z)$ (where ϵ is a small parameter characterizing the strength of the perturbation), and the rest can be related to $f_0(x)$ by conjugation [to $\mathscr{O}(\epsilon)$ accuracy]. The perturbation $\epsilon\mathscr{G}(f_0(x))$ gives the leading order correction $-\epsilon d[P_0(x)\mathscr{G}(x)]/dx$ to the unperturbed probability distribution $P_0(x)$. This correction term has an odd symmetry as contrasted with the even symmetry of $P_0(x)$ and of the correction coming from conjugation.[(5)] It is found that the LCE has no correction of $\mathscr{O}(\epsilon)$ and that there exists a negative upper bound, namely, $-2\epsilon^2P_0^2(1/2)\mathscr{G}^2(1/2)$ for its second-order

[3] The symmetry properties often discussed throughout the paper are always to be understood with respect to the center of the attractor, which will be the point $1/2$ in our coordinate system.

correction. This is in accord with the expectation that a doubly symmetric map has the maximum value for the LCE.[11,13,17] The two parts of the perturbation play quite different roles in the correlation function of two orbits separated by τ iterations. The perturbation $\epsilon\mathscr{G}(f_0(x))$ gives corrections for $\tau \geqslant 1$ but no correction for $\tau = 0$ to $\mathscr{O}(\epsilon)$ in contrast to conjugation, which gives correction only at $\tau = 0$.

In all of our examples the unperturbed map is the logistic parabola in the fully developed chaotic state, i.e., $f_0(x) = 1 - (2x-1)^2$. The main family of maps for which we have applied our procedure consists of polynomials of $(2x-1)^2$. We have treated in detail the fourth degree polynomial map $f(x) = 1 - (1-\epsilon)(2x-1)^2 - \epsilon(2x-1)^4$ and have found $-\epsilon^2/16 + \mathscr{O}(\epsilon^3)$ as a correction to the unperturbed LCE. The most notable finding regarding the correlation function of the quartic map is that to $\mathscr{O}(\epsilon)$ the correlation extends only to one iteration and to $\mathscr{O}(\epsilon^2)$ to two iterations. Investigations to first order in perturbation theory have been extended to polynomials of higher degree.

We have examined fully developed chaotic behavior at the $1 \to 2$ and $2 \to 4$ band splitting points and at the crisis point where the window associated with the period-three orbit ends. Our theoretical results agree well with numerical ones.[10,11,18] We use also our results for the polynomial maps to deduce an approximate theoretical value of the LCE for the universal chaos function, i.e., for the amplitude in the Huberman–Rudnick scaling law.[19,12]

The paper is organized as follows. In Section 2 the general framework of our investigations is set up. In Section 3 a symmetry condition is established for the distribution function of symmetric maps. Section 4 is devoted to the perturbation theory and to its application to polynomial maps. The first terms of the series expansion for the LCE and the correlation function are calculated in Section 5. We outline an iterative method for the measure in Section 6 and compare numerical and perturbation theoretical results. The map $1 - |2x-1|^{2(1+\epsilon)}$ is considered in Section 7.

2. FULLY DEVELOPED CHAOS

In the present work we focus our attention on maps of the interval $[0, 1]$ onto itself

$$x_{n+1} = f(x_n) \tag{2.1}$$

where $f(0) = f(1) = 0$ and $f(x)$ is supposed to have a single kth-order maximum at $\hat{x}$, where $f(\hat{x}) = 1$ (see Fig. 1). Further specifications understood in the definition of $f(x)$ are that it is differentiable, except possibly at

$\hat{x}$, its slope at $x = 0$ is greater than one, i.e., $f'(0) > 1$,[4] and it is monotonically increasing and decreasing for $x < \hat{x}$ and $x > \hat{x}$, respectively. We assume that the map (2.1) does not have any stable periodic orbit, but exibits chaotic behavior, which is fully developed in the sense that the iterations starting from almost any initial value fill out almost the whole interval (0, 1). This is the case, for example, if the map (2.1) is everywhere expanding or if $1/[|f'(x)|]^{1/2}$ is a convex function, which means that its Schwarzian derivative is negative.(1,6,7)

An important feature of such chaotic iterations is ergodicity, which means, that for almost any initial value the average of a function $h(x)$ can be expressed as

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} h(x_n) = \int_0^1 P_f(x) h(x)\, dx \tag{2.2}$$

where $P_f(x)$ is the stationary probability distribution function, describing the iteration (2.1). Thus

$$\int_0^1 h(x) P_f(x)\, dx = \int_0^1 h(f(x)) P_f(x)\, dx \tag{2.3}$$

This equation can be considered as the requirement for the stationarity or the invariance of the probability distribution function. A differential form of the invariance condition is also known, namely,

$$P_f(y)\, dy = P_f(x_0)\, dx_0 + P_f(x_1)\, dx_1 \tag{2.4}$$

where $f(x_0) = f(x_1) = y$, and the differentials are taken so that $dy = |f'(x_0)|\, dx_0 = |f'(x_1)|\, dx_1$ (see, e.g., Ref. 17). Equation (2.4) clearly demonstrates the stationarity: if points are distributed according to $P_f(x)$ then the distribution does not change after iterations.

Now we propose an alternative expression of the invariance by integrating both sides of (2.4)

$$\mu_f(y) = 1 + \mu_f\big(f_l^{-1}(y)\big) - \mu_f\big(f_u^{-1}(y)\big) \tag{2.5}$$

where

$$\mu_f(y) = \int_0^y P_f(x)\, dx \tag{2.6}$$

is the invariant measure.[5] Furthermore $f_l^{-1}(y) < \hat{x}$ and $f_u^{-1}(y) > \hat{x}$ denote the lower and upper branches of the inverse of the function $f(x)$, respec-

[4] The derivative will be denoted by a prime throughout this paper.

[5] It is, of course, a probabilistic measure absolutely continuous with respect to the Lebesgue measure. Since we will be interested only in such measures, these properties will be understood in the following.

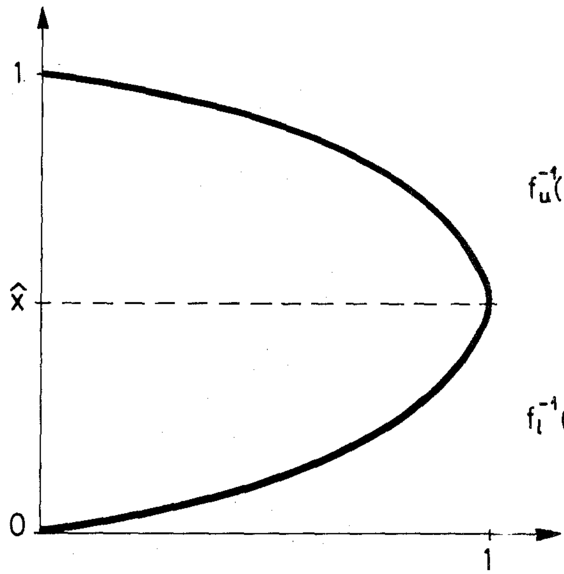

Fig. 2. The upper and lower branches of the inverse of a single-humped map generating fully developed chaos.

tively (see Fig. 2). Equation (2.5) is our basic relation for both perturbative and numerical considerations.

The LCE characterizing ergodic iterations is

$$\lambda_f = \int_0^1 P_f(x) \ln |f'(x)| \, dx \tag{2.7}$$

If the LCE is positive, nearby trajectories diverge with the average characteristic time $1/\lambda_f$, so the LCE measures the sensitive dependence on the initial condition.

When a map exhibiting fully developed chaos and the corresponding probability distribution function are both symmetric we will call the map doubly symmetric. Now we show that for a doubly symmetric $f(x)$, $\lambda_f = \ln 2$. In other words, the degree of irregularity is the same for such processes. Using the invariance condition (2.3) we have

$$\begin{aligned}\lambda_f &= \int_0^1 P_f(x) \ln |f'(x)| \, dx \\ &= \int_0^1 P_f(x) \ln |P_f(f(x)) \, df / P_f(x) \, dx| \, dx\end{aligned}$$

It is obvious from (2.4) that if $f(x)$ is doubly symmetric

$$P_f(f(x))\,|df(x)| = 2P_f(x)\,dx$$

and the above statement follows.

A further important quantity characterizing chaotic processes is the correlation function defined as

$$C_f(\tau) = \int_0^1 (x - \bar{x})\left[f^{(\tau)}(x) - \bar{x}\right]P_f(x)\,dx, \qquad \bar{x} = \int_0^1 xP_f(x)\,dx \quad (2.8)$$

where $f^{(\tau)}(x)$ is the τth iterate of $f(x)$ $[f^{(0)}(x) = x]$. Another consequence of the double symmetry is the noncorrelated behavior of subsequent iterations. As can be easily seen in this case[5]

$$C_f(\tau) = C(0)\delta_{\tau 0}$$

where $\delta_{\tau 0} = 1$ if $\tau = 0$ and it is zero elsewhere.

Finally we cite some properties of conjugation[5] needed in the following. The map $g(x)$ is conjugate to $f(x)$ if there exists a continuous, smooth, invertible function $u(x)$, for which

$$g(x) = u\left(f\left(u^{-1}(x)\right)\right) \quad (2.9)$$

The probability distribution for the map $g(x)$ can be expressed as

$$P_g(x) = P_f\left(u^{-1}(x)\right)\frac{du^{-1}(x)}{dx} \quad (2.10)$$

An important feature of conjugation is related to the problem of symmetry: if $f(x)$ is a doubly symmetric map, then every symmetric map conjugate to $f(x)$ is also a doubly symmetric one.[5]

3. ASYMPTOTIC BEHAVIOR OF THE INVARIANT MEASURE. A SYMMETRY CONDITION

Let us consider the asymptotic behavior of the invariant measure of symmetric maps of type (2.1) near the points $x = 0$ and $x = 1$. Equation (2.5) can be written now as

$$\mu_f(f(x)) = 1 + \mu_f(x) - \mu_f(1 - x), \qquad 0 \leqslant x \leqslant 1/2 \quad (3.1)$$

(For $1/2 \leqslant x \leqslant 1$ the signs of the second and third term on the right-hand side change.) Assume the map $f(y)$ has a maximum of kth order, i.e., for $y \approx \hat{x} = 1/2$ the map takes the form $f(y) \approx 1 - a|y - 1/2|^k$. Denoting $a|y - 1/2|^k$ by x and applying (3.1) and (2.6) we get

$$\mu_f(1 - x) \approx 1 - 2P_f(1/2)(x/a)^{1/k}, \qquad x \ll 1 \quad (3.2)$$

For a measure not differentiable at $\hat{x} = 1/2$

$$2P_f(1/2) = \mu_f'(1/2+0) + \mu_f'(1/2-0)$$

is taken.

Next we iterate the point $1 - x$. By writing

$$f(1-x) = f(x) \approx cx^j, \qquad x \ll 1 \tag{3.3}$$

and assuming that the asymptotic form of the invariant measure is

$$\mu_f(x) \approx bx^l, \qquad x \ll 1 \tag{3.4}$$

we get by using (3.1), (3.2), (3.3), and (3.4)

$$bc^l x^{lj} \approx bx^l + 2P_f(1/2)(x/a)^{1/k}, \qquad x \ll 1 \tag{3.5}$$

The dominant term on the right-hand-side must asymptotically equal the left-hand-side. Comparing both the exponents and the amplitudes for $j < 1$ one has

$$l = 1/kj \tag{3.6a}$$

and

$$b = \frac{2P_f(1/2)}{c^{1/(kj)}a^{1/k}} \tag{3.6b}$$

If $j = 1$ one should distinguish the cases $c > 1$ and $c = 1$. In the former one

$$l = 1/k \tag{3.7a}$$

$$b = \frac{2P_f(1/2)}{a^{1/k}(c^{1/k} - 1)} \tag{3.7b}$$

Note that for $j = 1$, c is the initial slope, $c = f'(0)$. If $f'(0) = 1$, however, we only know that

$$l < 1/k \tag{3.8}$$

and b cannot be determined at this stage. Finally, $j > 1$, or $j = 1$ with $c < 1$ are not allowed: in these cases the map would not produce fully developed chaos.

By means of the above results a symmetry condition can easily be deduced for the probability distribution function by using (2.6). Namely, by comparing (3.2) with (3.4), (3.6), (3.7), (3.8) we see that its asymptotic behavior near 0 and 1 is the same only if

$$j = 1, \qquad f'(0) = 2^k \tag{3.9}$$

This is, of course, a necessary condition for the odd symmetry of the invariant measure. The result (3.9) has been derived under the conditions that $P_f(1/2)$ and $P_f^{-1}(1/2)$ are finite, which is the typical case. The generalization is straightforward but is not considered here.

Consider, for example, the map

$$f(x) = 1 - |1 - 2x|^k \tag{3.10}$$

which for $k > 1/2$ exhibits fully developed chaos.[6] The symmetry condition (3.9) is fulfilled by (3.10) only for the piecewise linear map ($k = 1$) and for $k = 2$, in which case (3.10) is the logistic map in the fully developed chaotic state.

4. PERTURBATION THEORY AND ITS APPLICATION TO POLYNOMIAL MAPS

Starting from equation (2.5) for the invariant measure we develop a perturbative approach to the probability distribution function for symmetric maps.[7]

Let us consider the symmetric map $f(\epsilon, x)$, where ϵ is a small parameter. Assume that the probability distribution function $P_0(x)$ belonging to the zeroth-order map $f(0, x) \equiv f_0(x)$ is known. Furthermore, we denote the perturbed invariant measure by $\mu(\epsilon, x)$, whereas $\mu(0, x) \equiv \mu_0(x)$ and $\mu_0'(x) = P_0(x)$. As shown in Section 2 $\mu(\epsilon, x)$ can be determined from

$$\mu(\epsilon, f(\epsilon, x)) = 1 + \mu(\epsilon, x) - \mu(\epsilon, 1 - x), \qquad 0 \leqslant x \leqslant 1/2 \tag{4.1}$$

We consider the maps

$$\begin{aligned} f(\epsilon, x) &= f_0(x) = \epsilon f_1(x) + \epsilon^2 f_2(x) + \cdots \\ f_k(x) &= f_k(1 - x) \\ f_k(0) &= f_k(1/2) = 0, \qquad k \geqslant 1 \end{aligned} \tag{4.2}$$

and suppose that the measure and the probability distribution can be expanded as

$$\begin{aligned} \mu(\epsilon, x) &= \mu_0(x) + \epsilon\mu_1(x) + \epsilon^2\mu_2(x) + \cdots \\ \mu_k(0) &= \mu_k(1) = 0, \qquad k \geqslant 1 \end{aligned} \tag{4.3}$$

$$P(\epsilon, x) = \partial\mu(\epsilon, x)/\partial x = P_0(x) + \epsilon P_1(x) + \epsilon^2 P_2(x) + \cdots \tag{4.4}$$

[6] There exists an invariant measure for this map when $k > 1/2$ which is unique and ergodic. Namely, in the region $1/2 < k \leqslant 1$ the map is everywhere expanding and then the theorem by Li and Yorke (see, e.g., Ref. 1) applies. For $k > 1$ the map has a negative Schwarzian derivative and fulfills also the other conditions under which Misiurewicz proved the existence of an absolutely continuous invariant measure.[6,7] For $k = 1/2$ see Section 7.

[7] The generalization of the perturbation theory for nonsymmetric perturbations is, in principle, straightforward. Since it is not needed in the present work we will not deal with this more general case.

Further assuming that $\mu_k(f(\epsilon, x))$, with $f(\epsilon, x)$ given by (4.2), can also be expanded in powers of ϵ, Eq. (4.1) yields

$$\mu_k(f_0(x)) = F_k(f_0(x)) + \mu_k(x) - \mu_k(1-x), \qquad 0 \leqslant x \leqslant 1/2 \quad (4.5)$$

Here $F_0(x) \equiv 1$ and for $k \geqslant 1$ $F_k(x)$ is given in terms of $f_0, f_1, \ldots, f_k$ and $\mu_0, \mu_1, \ldots \mu_{k-1}$ in the Appendix.

It is easy to show that if the measure $\mu_0(x)$ is unique for the map $f_0(x)$, which is, of course, our basic assumption, then the solution of (4.5), if it exists, is also unique. Suppose that $\mu_0, \mu_1, \ldots, \mu_{k-1}$ are unique; then $F_k(x)$ is unique, too. If (4.5) has two solutions then their difference $\Delta\mu(x)$ satisfies the equation

$$\Delta\mu(f_0(x)) = \Delta\mu(x) - \Delta\mu(1-x), \qquad 0 \leqslant x \leqslant 1/2 \quad (4.6)$$

From the uniqueness of the solution of (4.5) for $k = 0$ follows, however, that (4.6) has the only solution $\Delta\mu(x) \equiv 0$. This way, the uniqueness can be proven for all indices $k \geqslant 1$.

According to (A.3) the equation for the first-order correction $\mu_1(x)$ takes the form

$$\mu_1(f_0(x)) = -f_1(x)P_0(f_0(x)) + \mu_1(x) - \mu_1(1-x), \qquad 0 \leqslant x \leqslant 1/2 \quad (4.7)$$

It is useful to compare Eq. (4.7) with the result of conjugation to $f_0(x)$ with a conjugating function that deviates from the identity by a term of $\mathscr{O}(\epsilon)$, i.e.,

$$u(x) = x + \epsilon\mathscr{H}(x) + \mathscr{O}(\epsilon^2) \quad (4.8)$$

$$\mathscr{H}(x) = -\mathscr{H}(1-x) \quad (4.9)$$

The symmetry property of $\mathscr{H}(x)$ follows from the requirement that both the initial and the conjugate maps be symmetric. By writing the conjugated function and the corresponding probability distribution as

$$f^c(\epsilon, x) = f_0(x) + \epsilon f_1^c(x) + \mathscr{O}(\epsilon^2) \quad (4.10)$$

and

$$P^c(\epsilon, x) = P_0(x) + \epsilon P_1^c(x) + \mathscr{O}(\epsilon^2) \quad (4.11)$$

respectively, we obtain by taking into account Eqs. (2.9), (2.10), (4.8), (4.9), (4.10), and (4.11)

$$f_1^c(x) = \mathscr{H}(f_0(x)) - \mathscr{H}(x)f_0'(x) \quad (4.12)$$

and

$$P_1^c(x) = -\left[\mathscr{H}(x)P_0(x)\right]' \quad (4.13)$$

Equation (4.13) yields for the measure

$$\mu^c(\epsilon, x) = \mu_0(x) + \epsilon\mu_1^c(x) + \mathcal{O}(\epsilon^2) \tag{4.14}$$

with

$$\mu_1^c(x) = -\mathcal{H}(x)P_0(x) \tag{4.15}$$

On the other hand, looking for a solution of Eq. (4.7) in the form of (4.15), (4.9) we arrive at (4.12) as a condition for $f_1(x)$ by using Eq. (2.4) for $P_0(x)$.

We proceed by exploiting the fact that a general solution can be separated in the following way:

$$\mu_1(x) = -P_0(x)\mathcal{G}(x) - P_0(x)\mathcal{H}(x) \tag{4.16}$$

$$\mathcal{G}(x) = \mathcal{G}(1 - x) \tag{4.17}$$

and $\mathcal{H}(x)$ has the odd symmetry (4.9). For a given $\mu_1(x)$ and $P_0(x)$ the functions $\mathcal{G}(x)$ and $\mathcal{H}(x)$ are uniquely specified by these equations. We define now $f_1^c(x)$ by Eq. (4.12), and $f_1^{nc}(x)$ by

$$f_1(x) = f_1^{nc}(x) + f_1^c(x) \tag{4.18}$$

Substituting (4.16), (4.17), (4.18), (4.12) into Eq. (4.7) and using Eq. (2.4) for $P_0(x)$ we obtain

$$f_1^{nc}(x) = \mathcal{G}(f_0(x)) - \mathcal{G}(x)f_0'(x)\frac{P_0(x) - P_0(1-x)}{P_0(x) + P_0(1-x)} \tag{4.19}$$

If $P_0(x)$ is symmetric, which is the most important case concerning applications, it follows that

$$f_1^{nc}(x) = \mathcal{G}(f_0(x)) \tag{4.20}$$

It means that in this case there should always exist a splitting (4.18) for a perturbation $f_1(x)$ in such a way that to $\mathcal{O}(\epsilon)$ $f_0(x) + \epsilon f_1^c(x)$ is conjugate to $f_0(x)$ and $f_1^{nc}(x)$ has the properties specified by (4.20), (4.17). The assumption leading to this conclusion is that (4.7) has a solution.

To use $\mathcal{G}$ and $\mathcal{H}$ for specifying the perturbation will be often advantageous in the following. Here we mention as an example that the change in the average value of the quantity $A(x)$,

$$\delta A = \int_0^1 A(x)[P(x) - P_0(x)]\,dx$$

can be expressed to $\mathcal{O}(\epsilon)$ as

$$\delta A = \epsilon\int_0^1 A'(x)[\mathcal{G}(x) + \mathcal{H}(x)]P_0(x)\,dx$$

We can consider now the perturbation expansion for the polynomial maps

$$f(x) = f_0(x) + \epsilon f_1(x) \tag{4.21}$$

$$f_0(x) = 1 - (1 - 2x)^2 = 4x(1 - x) \tag{4.22}$$

$$\epsilon f_1(x) = \sum_{i=1}^{n} \alpha_i (1 - 2x)^{2i}, \qquad \sum_{i=1}^{n} \alpha_i = 0 \tag{4.23}$$

Note that $f_1(x)$ contains the factor $(1-2x)^2[1-(1-2x)^2] = f_0(x)[1 - f_0(x)]$ due to the boundary conditions (4.2). The unperturbed map is the logistic parabola in the fully developed chaotic state, which has the invariant measure[20,17]

$$\mu_0(x) = (2/\pi)\arcsin\sqrt{x} \tag{4.24}$$

and the probability distribution function

$$P_0(x) = \frac{1}{\pi[x(1-x)]^{1/2}} \tag{4.25}$$

In the following the most important case will be the quartic map

$$f(x) = 1 - (1 - \epsilon)(1 - 2x)^2 - \epsilon(1 - 2x)^4 \tag{4.26}$$

corresponding to the choice $\epsilon = \alpha_1 = -\alpha_2$, $\alpha_i = 0$, $i \geqslant 3$. It can easily be shown that the quartic map has a negative Schwarzian derivative and that its fixed point at $x = 0$ is unstable for $\epsilon > -3/4$. It also satisfies the other conditions in the range $-3/4 < \epsilon \leqslant 1$ sufficient to have an absolutely continuous invariant measure according to the results by Misiurewicz; this measure is unique, ergodic, and its support is the $[0, 1]$ interval.[1,6,7] The boundaries in the parameter space within which these conditions are met will not be given here for the higher-degree polynomial maps. In our applications all the parameters α_i are small, in which case they are obviously fulfilled.

Using the fact that according to (4.22), (4.25) $P_0(x) = 2P_0(f_0(x))\,|1-2x|$, the solution of (4.7), with $f_1(x)$ given by (4.23), can easily be found in the form $\mu_1(x) = \nu_1(n; (1-2x))P_0(x)$. Here and in (4.31) below $\nu(q; z)$ denotes a qth-degree polynomial of z satisfying $\nu(q; 1) = \nu(q; -1) = 0$, i.e., containing the factor $(1 - z^2)$. Separating ν_1 into its symmetric and antisymmetric parts we obtain the functions $\mathscr{G}$ and $\mathscr{H}$ defined by (4.16), (4.17), and (4.9) as

$$\mathscr{G}(\zeta; x) = x(1-x) \sum_{i=0}^{k} \zeta_i x^i (1-x)^i \tag{4.27}$$

$$\mathscr{H}(\eta; x) = (1-2x) \sum_{i=1}^{l} \eta_i x^i (1-x)^i \tag{4.28}$$

Here $k = l = (n-2)/2$ if n is even and $k = (n-3)/2$, $l = (n-1)/2$ if n is odd. The parameters ζ_i, η_i can be expressed in terms of $\alpha_i - s$. As an example we give them for $n = 8$

$$\alpha_8 = \epsilon\zeta_3 \tag{4.29a}$$

$$\alpha_7 = -2\epsilon(\eta_3 + 2\zeta_3) \tag{4.29b}$$

$$\alpha_6 = \epsilon(-\zeta_2 + 6\zeta_3 + 7\eta_3) \tag{4.29c}$$

$$\alpha_5 = \epsilon(3\zeta_2 - 4\zeta_3 + 2\eta_2 - 9\eta_3) \tag{4.29d}$$

$$\alpha_4 = \epsilon(\zeta_1 - 3\zeta_2 + \zeta_3 - 5\eta_2 + 81\eta_3/16) \tag{4.29e}$$

$$\alpha_3 = \epsilon(-2\zeta_1 + \zeta_2 - 2\eta_1 + 15\eta_2/4 - 19\eta_3/16) \tag{4.29f}$$

$$\alpha_2 = \epsilon(-\zeta_0 + \zeta_1 + 4\eta_1 - \eta_2/2 + 3\eta_3/16) \tag{4.29g}$$

$$\alpha_1 = \epsilon(\zeta_0 - 2\eta_1 - \eta_2/4 - \eta_3/16) \tag{4.29h}$$

Substituting (4.21)–(4.25), (4.16), (4.27), and (4.28) into (A.3) and using the fact, that $P_0'(x) = -P_0(x)(1-2x)[x(1-x)]^{-1}/2$ we arrive at

$$F_2(f_0(x)) = -E_2(2n-1;(1-2x)^2)P_0(f_0(x)) \tag{4.30}$$

where $E(q;z)$ denotes a qth degree polynomial of z satisfying $E(q;0) = E(q;1) = 0$, i.e., containing a factor $z(z-1)$. Then we can solve Eq. (4.5) for $\mu_2(x)$ in the same way as we have done for $\mu_1(x)$; consequently the solution has the form

$$\mu_2(x) = \nu_2(2n-1;(1-2x))P_0(x) \tag{4.31}$$

In the case of the quartic map (4.26) we obtain for the second-order correction for the measure

$$\mu_2(x) = \tfrac{3}{16}\left[1-(1-2x)^2\right]\left[1+(1-2x)\right]P_0(x) \tag{4.32}$$

and for the probability distribution function using (4.25), (4.16), (4.27)–(4.29)

$$P(x) = \frac{1}{\pi\left[x(1-x)\right]^{1/2}}\left[1+\epsilon\left(x-\frac{1}{2}\right)+\epsilon^2\frac{3}{4}(1-5x+4x^2)\right]+\mathscr{O}(\epsilon^3) \tag{4.33}$$

Notice that the initial slope of the quartic map is $4(1+\epsilon)$ and consequently the symmetry condition (3.9) is satisfied only for $\epsilon = 0$.

It can be shown that for polynomial maps also the higher-order corrections to the measure have the form of a product of $P_0(x)$ with a polynomial of $(1-2x)$. We are not going to discuss these higher-order corrections in the present paper, but turn to the evaluation of the low-order corrections to the LCE and to the correlation function.

5. THE LYAPUNOV CHARACTERISTIC EXPONENT AND THE CORRELATION FUNCTION

Let us consider first the perturbation expansion for the LCE. We substitute (4.2) and (4.4) into (2.7) and assume that the resulting expression can be expanded as

$$\lambda(\epsilon) = \lambda_0 + \epsilon\lambda_1 + \epsilon^2\lambda_2 + \cdots \tag{5.1}$$

$$\lambda_0 = \int_0^1 P_0(x)\ln|f_0'(x)|\,dx \tag{5.2}$$

$$\lambda_1 = \int_0^1 P_1(x)\ln|f_0'(x)|\,dx + \int_0^1 P_0(x)\left[f_1'(x)/f_0'(x)\right]dx \tag{5.3}$$

Taking into account (4.16), (4.18), (4.12), (4.19) the terms in (5.3) containing $\mathscr{H}$ cancel, which corresponds to the fact that conjugation does not change the LCE, and λ_1 can be expressed in terms of $\mathscr{G}$ as

$$\lambda_1 = -\int_0^1\left[\frac{P_0(x) - P_0(1-x)}{P_0(x) + P_0(1-x)}\right]'\mathscr{G}(x)P_0(x)\,dx \tag{5.4}$$

which yields $\lambda_1 = 0$ in the case of a symmetric $P_0(x)$.

Moreover, we can obtain a useful negative upper limit for the second-order correction λ_2 when the zeroth-order map exhibits double symmetry. For this purpose we start from the inequality[13] for the LCE

$$\lambda \leqslant L \tag{5.5}$$

where L for maps generating fully developed chaos takes the form

$$L = -M\ln M - (1-M)\ln(1-M) \tag{5.6}$$

Here

$$M = \int_0^{1/2} P(x)\,dx = \mu(1/2) \tag{5.7}$$

may be called the asymmetry number of the probability distribution function. Note that for symmetric distribution functions $M = 1/2$. Using (4.16), (4.17), (4.9) and $\mu_0(1/2) = 1/2$ we get

$$L(\epsilon) = \ln 2 - \epsilon^2 2P_0^2(1/2)\mathscr{G}^2(1/2) + \mathscr{O}(\epsilon^3) \tag{5.8}$$

Taking into account that $\lambda_0 = \ln 2$ for a symmetric $P_0(x)$ (see Section 2) we obtain

$$\lambda_2 \leqslant -2P_0^2(1/2)\mathscr{G}^2(1/2) \tag{5.9}$$

A negative upper limit for λ_2 is in accord with the fact that $\ln 2$ represents the possible maximum value for the LCE.[13]

Now we calculate the corrections to the LCE in the case of the quartic map (4.26). Here according to (5.4) $\lambda_1 = 0$ and for the second-order correction we get

$$\lambda_2 = \int_0^1 P_2(x)\ln|f_0'(x)|\,dx - \frac{1}{2}\int_0^1 P_0(x)\big[f_1'(x)/f_0'(x)\big]^2\,dx$$
$$+ \int_0^1 P_1(x) f_1'(x)/f_0'(x)\,dx = 1/16 \tag{5.10}$$

where (4.26) and (4.33) have been used. Hence, the expansion of the LCE starts as

$$\lambda(\epsilon) = \ln 2 - \frac{\epsilon^2}{16} + \mathscr{O}(\epsilon^3) \tag{5.11}$$

Furthermore using (4.33), we obtain from (5.6), (5.7)

$$L(\epsilon) = \ln 2 - \frac{\epsilon^2}{2\pi^2} + \frac{3\epsilon^3}{4\pi^2} + \mathscr{O}(\epsilon^4) \tag{5.12}$$

for the quartic map. It is apparent that the perturbation theory is in accordance with the inequality (5.5).

Concerning the polynomial maps (4.21), (4.22), and (4.23) the insertion of (4.25) and (4.27) into (5.8) yields

$$L(\epsilon; \eta, \zeta) = \ln 2 - \frac{\epsilon^2}{2\pi^2}\left[\sum_{i=0}^{k} 4^{-i}\zeta_i\right]^2 + \mathscr{O}(\epsilon^3) \tag{5.13}$$

Note, that to $\mathscr{O}(\epsilon^2)$ the parameters η_i do not appear.

We turn now to the evaluation of the correlation function. In the general case only corrections of first order are considered. Substituting into Eq. (2.8) the expansion (4.4) and

$$\bar{x} = \frac{1}{2} + \epsilon\int_0^1 x P_1(x)\,dx \tag{5.14}$$

we obtain to $\mathscr{O}(\epsilon)$ the expression

$$C(\epsilon,\tau) = \int_0^1\left(x - \frac{1}{2}\right)\big[f^{(\tau)}(\epsilon,x) - \bar{x}\big]P_0(x)\,dx$$
$$- \epsilon\int_0^1 x P_1(x)\,dx \int_0^1\left[f_0^{(\tau)}(x) - \frac{1}{2}\right]P_0(x)\,dx$$
$$+ \epsilon\int_0^1\left(x - \frac{1}{2}\right)\left[f_0^{(\tau)}(x) - \frac{1}{2}\right]P_1(x)\,dx + \mathscr{O}(\epsilon^2) \tag{5.15}$$

We consider here the case when $P_0(x)$ is symmetric. Recalling that we assume both $f_0(x)$ and $f(\epsilon, x)$ to be symmetric, it follows immediately that the first integral in (5.15) vanishes for $\tau > 0$ and the second one turns out to be zero for any τ value by applying (2.3).

According to Eqs. (2.6), (4.16), (4.13) we can write

$$P_1(x) = P_1^c(x) + P_1^{nc}(x) \tag{5.16}$$

where now $P_1^c(x)$ is symmetric and $P_1^{nc}(x)$ is antisymmetric. Accordingly (5.15) leads to

$$C(\epsilon, \tau) = C_0(\tau) + \epsilon C_1^c(\tau) + \epsilon C_1^{nc}(\tau) + \mathscr{O}(\epsilon^2) \tag{5.17}$$

where

$$C_0(\tau) = \delta_{\tau 0} \int_0^1 \left(x - \frac{1}{2}\right)^2 P_0(x)\, dx \tag{5.18a}$$

$$C_1^c(\tau) = \delta_{\tau 0} \int_0^1 \left(x - \frac{1}{2}\right)^2 P_1^c(x)\, dx \tag{5.18b}$$

$$C_1^{nc}(\tau) = \int_0^1 \left(x - \frac{1}{2}\right)\left[f_0^{(\tau)}(x) - \frac{1}{2}\right] P_1^{nc}(x)\, dx \tag{5.18c}$$

The $\delta_{\tau 0}$ character of $C_0(\tau)$ and $C_1^c(\tau)$ is a natural consequence of the fact that $f_0(x)$ is doubly symmetric and that conjugation does not alter this feature (see Section 2). Notice that $C_1^{nc}(\tau) = 0$ for $\tau = 0$ is a result which could not be foreseen.

Let us consider the map (4.21)–(4.23). The unperturbed map is (4.21b) for which a correlation function was calculated by Grossmann and Thomae[5]:

$$C_0(\tau) = \tfrac{1}{8}\delta_{\tau 0} \tag{5.19}$$

Using (4.16), (4.27), (4.28), (4.29), and (5.18) we get for $n = 8$, i.e., for the 16th degree polynomial map

$$C_1^c(\eta; \tau) = -\delta_{\tau 0}(2^{-5} \cdot \eta_1 + 2^{-8} \cdot \eta_2 + 5 \cdot 2^{-13} \cdot \eta_3) \tag{5.20a}$$

$$\begin{aligned} C_1^{nc}(\zeta; \tau) = {} & \delta_{\tau 1}(-2^{-5} - \zeta_0 + 5 \cdot 2^{-13} \cdot \zeta_2 + 7 \cdot 2^{-15} \cdot \zeta_3) \\ & + \delta_{\tau 2}(3 \cdot 2^{-9} \cdot \zeta_1 + 5 \cdot 2^{-12} \cdot \zeta_2 + 7 \cdot 2^{-15} \cdot \zeta_3) \\ & + \delta_{\tau 3} \cdot 7 \cdot 2^{-17} \cdot \zeta_3 \end{aligned} \tag{5.20b}$$

Concerning the quartic map ($\eta_i = \zeta_i = 0,\ i \geqslant 1$) the most striking feature of our finding for the first-order correction is that it extends only to one step. In this case we have carried out a second-order calculation, too, yielding the following correlation function:

$$\begin{aligned} C(\epsilon, \tau) = {} & (\tfrac{1}{8} + 2^{-7} \cdot \epsilon^2)\delta_{\tau 0} + (-2^{-5} \cdot \epsilon + 3 \cdot 2^{-7} \cdot \epsilon^2)\delta_{\tau 1} \\ & + 2^{-6} \cdot \epsilon^2 \cdot \delta_{\tau 2} + \mathscr{O}(\epsilon^3) \end{aligned} \tag{5.21}$$

6. AN ITERATIVE APPROACH TO THE INVARIANT MEASURE. COMPARISON OF THEORETICAL AND NUMERICAL RESULTS

In this section we introduce an effective numerical technique for computing stationary distribution functions. There are several approaches for calculating probability distribution functions.[5,13,20–22] For the invariant measure we propose a modified version of the functional iteration with the Frobenius–Perron operator (for the Frobenius–Perron operator see, e.g., Ref. 5). That means we investigate the integrals of the distribution functions instead of the probability distributions themselves. If the invariant measure is unique and stable under perturbations then it can be reached from the initial function $\mu_f^{(0)}(x) \equiv x$ by the iterative procedure

$$\mu_f^{(n)}(x) = \mu_f^{(n-1)}\big(f_l^{-1}(x)\big) - \mu_f^{(n-1)}\big(f_u^{-1}(x)\big) + 1 \tag{6.1}$$

The fixed point of (6.1) satisfies (2.5) and thus is the invariant measure $\mu_f(x)$. The general form of the nth functional iterate is expressed as

$$\mu_f^{(n)}(x) = 1 + \sum_{a_0, \ldots, a_{n-1} = l,u} (-1)^A f_{a_0}^{-1}\big(f_{a_1}^{-1}\big(\cdots \big(f_{a_{n-1}}^{-1}(x)\big) \cdots \big)\big) \tag{6.2}$$

where

$$A = \sum_{i=0}^{n-1} \delta_{ua_i}, \qquad \delta_{uu} = 1, \qquad \delta_{ul} = 0 \tag{6.3}$$

The iteration method can be used not only numerically but analytically as well if $f^{-1}(x)$ can be explicitly given. It can be easily seen that the method yields the correct exponents for the power law behavior near $x = 1$ (i.e., $1/k$ if $f(x)$ has a kth order maximum, according to (3.2)) already from the first iteration on and near $x = 0$ [i.e., $1/jk$, see (3.6a)] already from the second iteration on, if $j < k$.

We have applied the iteration method to compute the asymmetry number (5.7) for the quartic map (4.26). The result can be compared with that of our perturbation expansion

$$M(\epsilon) = \frac{1}{2} - \frac{\epsilon}{2\pi} + \frac{3\epsilon^2}{8\pi} + \mathcal{O}(\epsilon^3) \tag{6.4}$$

obtained with the help of (4.33). Figure 3 shows that the theoretical parabola fits well even for relatively large $|\epsilon|$ values.

As an example for polynomial maps we investigate the logistic map

$$f_L(r, x) = rx(1 - x) \tag{6.5}$$

at parameter values $r = \rho_1, \rho_2, \tilde{\rho}_3$ corresponding to the band splitting points $1 \to 2, 2 \to 4$ and to the crisis point where the window associated with the period three orbit ends, respectively. These parameter values are $\rho_1 =$

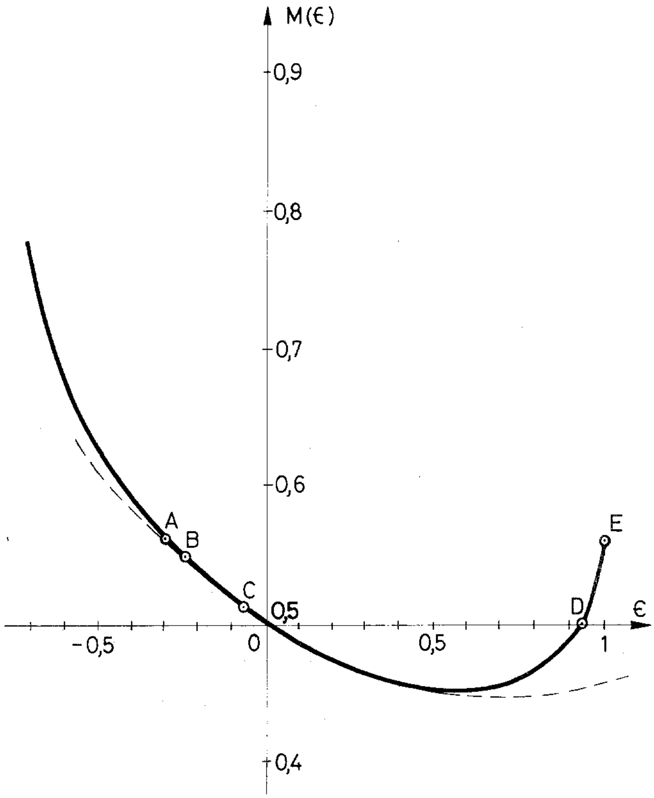

Fig. 3. The asymmetry number M for quartic maps as a function of ϵ measuring th deviation of the map from the logistic one. The solid curve represents values obtained in th iterative way, whereas the dotted line is the perturbative approximation (6.4). The points A, I and C are related to the period-two, period-four, and period-three chaos in the logistic cas respectively. Although the probability distribution is not symmetric at $\epsilon = 0.9405\ldots$, th asymmetry number equals $1/2$ there; see mark D. At E the quartic map does not contain quadratic term.

Table I. Values of Parameters for the Map $f_{\mathrm{CB}}^{(4)}$

Parameter	
ϵ	$-0.2592\ldots$
η_1	$0.1090\ldots$
η_2	$-3.85\ldots\times 10^{-3}$
η_3	$4.769\ldots\times 10^{-6}$
ζ_1	$0.0729\ldots$
ζ_2	$-3.889\ldots\times 10^{-4}$
ζ_3	$9.13\ldots\times 10^{-8}$

$3.67857351\ldots$, $\rho_2 = 3.592572184\ldots$, $\tilde{\rho}_3 = 3.856800652\ldots$,.[10-12] Fully developed chaos appears at these points in each of the bands for the second, fourth, and the third iterates of the map (6.5), respectively. Concerning the central band (i.e., the band containing the $x = 1/2$ point) the maps in question are symmetric and are polynomials of fourth, sixteenth, and eighth degree, respectively. By appropriate rescaling and shifting the coordinate system they can be brought to the form (4.21)–(4.23) and will be denoted by $f_{\text{CB}}^{(2)}$, $f_{\text{CB}}^{(4)}$, and $f_{\text{CB}}^{(3)}$, respectively. $f_{\text{CB}}^{(2)}(x)$ is an example of the quartic map (4.26) with $\epsilon = -0.29559$. The meaning of ϵ that the slope of the map at the origin is $4(1+\epsilon)$ can be kept for the higher degree polynomial maps, too, by choosing ζ_0 defined by (4.27) equal to unity. Then the other parameters introduced in (4.27) and (4.28) can be determined with the help of (4.29a)–(4.29h). For the map $f_{\text{CB}}^{(3)}(x)$ the nonzero ones are as follows: $\epsilon = -0.07127$, $\eta_1 = -0.01056$, $\zeta_1 = -1.339 \times 10^{-4}$. Table I contains the corresponding values for $f_{\text{CB}}^{(4)}$.

Let us consider first the correlation function. It is apparent that the $\eta_i - s$ and $\zeta_i - s$ can be regarded as small parameters, and calculating up to second order in ϵ, it is consistent to take into account only η_1 in the $O(\epsilon)$ correction and to take $\eta_i = \zeta_i = 0$ in the terms of second order in ϵ. It means that for $f_{\text{CB}}^{(3)}$ and $f_{\text{CB}}^{(4)}$ we can apply to this accuracy a 6th degree polynomial. Introducing the notation

$$C(\tau) = C_0(\tau) + \sum_i C^{(i)}\delta_{\tau,i} \tag{6.6}$$

the values of $C^{(i)} - s$ calculated with the help of (5.20a), (5.20b) and (5.21) are given in Table II. The results are in good agreement with the numerical experiment by Thomae and Grossmann.[10] In particular, for the maps $f_{\text{CB}}^{(2)}$ and $f_{\text{CB}}^{(4)}$ one can read off a value of about 0.01 (slightly larger for $f_{\text{CB}}^{(2)}$) for the correlation function at $\tau = 1$ from Fig. 5 in Ref. 10.

Turning to the LCE, it is important that, according to (5.4), there is no $O(\epsilon)$ correction. Concerning the second-order terms we can take again

Table II. Parameters of the Correlation Function, the LCE-s, and the Values of L^a

	$C^{(0)}$	$C^{(1)}$	$C^{(2)}$	λ	L
$f_{\text{CB}}^{(2)}$	0.0007	0.0113	0.0013	0.6876	0.6866
$f_{\text{CB}}^{(4)}$	0.0008	0.0097	0.0011	0.6890	0.6884
$f_{\text{CB}}^{(\infty)}$	0.0015	0.0098	0.0011	0.6886	0.6882
$f_{\text{CB}}^{(3)}$	$2 \cdot 10^{-5}$	0.0023	0.0001	0.6928	0.6929

[a] For the definition of the functions $f_{\text{CB}}^{(2)}$, $f_{\text{CB}}^{(4)}$, and $f_{\text{CB}}^{(3)}$ consult text. $f_{\text{CB}}^{(\infty)}$ stands for the universal chaos function.

$\eta_i = \zeta_i = 0$ for the maps $f_{\rm CB}^{(3)}$ and $f_{\rm CB}^{(4)}$ and then (5.11) can be applied not only for $f_{\rm CB}^{(2)}$ but for them as well. Moreover, it is consistent to use the expression (5.12) for L including the third-order term for $f_{\rm CB}^{(3)}$ and $f_{\rm CB}^{(4)}$, too, since the $\eta_i - s$ do not occur in second order in ϵ, according to (5.13), and the $O(\epsilon^2)$ terms which would contain a factor of ζ_i are completely negligible as compared to the $O(\epsilon^3)$ contribution. The results for λ and L are given in Table II. Note that the LCE for the map (6.5) can be obtained by dividing the values in Table II by the number of bands, which are two, four, and three at ρ_1, ρ_2 and $\tilde{\rho}_3$, respectively. Our values for the LCE can be compared at ρ_1 with the result 0.342 and at $\tilde{\rho}_3$ with the result $(\ln 2)(1 - 4 \times 10^{-4})/3$ obtained by Kai and Tomita[18] and by Grebogi, Ott, and Yorke,[11] respectively. For the sake of comparison our result at the latter point can be written as $(\ln 2)(1 - 4.6 \times 10^{-4})/3$.

Similarly the expression (6.4) for the asymmetry number M can be used consistently for $f_{\rm CB}^{(4)}$ and $f_{\rm CB}^{(3)}$, too, since the $\eta_i - s$ do not contribute to $O(\epsilon)$ and the other modifications due to the $\eta_i - s$ and $\zeta_i - s$ up to second order are completely negligible. The corresponding points are marked in Fig. 3.

Following the central band of the map (6.5), one can find a 4^kth-degree polynomial map $f_{\rm CB}^{(2^k)}(x)$ exhibiting fully developed chaos at the $2^{k-1} \to 2^k$ band splitting point. In the limit $k \to \infty$ $f_{\rm CB}^{(2^k)}(x)$ becomes the universal chaos function.[13,15,16,23] In the Feigenbaum space of functions[24] it lies on the purely repelling line; in fact, it is the mirror image of Feigenbaum's universal function $g_0(x)$. Its LCE is the amplitude Λ in the Huberman–Rudnick scaling law[19] $\lambda(k) = \Lambda/2^k$, $k \to \infty$, where $\lambda(k)$ is the LCE at the $2^{k-1} \to 2^k$ band splitting point. It is expected (and heuristic arguments can be given for it) that in the maps $f_{\rm CB}^{(2^k)}(x)$ the higher powers of $(1 - 2x)$ occur with smaller prefactors (a tendency which has already been seen for $k = 2$). We assume that this remains the case for the universal chaos function and carry out the calculation again to second-order accuracy with the help of a 6th degree polynomial map. Using the result[23] that the slope of the universal chaos function at the origin is 2.94805 . . . , one obtains $\epsilon = -0.26299$. By fitting the universal chaos function[13] with a parabola near its maximum, the value of α_1 can be determined, which yields through (4.29h) $\eta_1 = 0.1112$. Note that both ϵ and η_1 obtained in this way are close to their values at the $2 \to 4$ band splitting point, which shows the consistency of our assumption. The different characteristics can be calculated from (5.20a), (5.20b), (5.21), (5.11), (5.12) and are given in Table II. The results for the LCE and for L can be compared with the numerical results by Chang and Wright.[13] They have found for the LCE the value 0.6867, which within the accuracy of their computation, agrees with L (note that the quantity R used in Ref. 13 is $L/\ln 2$ in our notation).

7. INVESTIGATION OF THE MAP $1 - |2x - 1|^k$

As a further example for the application of the perturbation theory let us consider the map (3.10) rewritten as

$$f(\epsilon, x) = 1 - |2x - 1|^{2(1+\epsilon)} \tag{7.1}$$

One can formally expand (7.1) as

$$f(\epsilon, x) = 1 - (2x-1)^2\Big[1 + \epsilon \ln(2x-1)^2 + (\epsilon^2/2)\big(\ln(2x-1)^2\big)^2 + \cdots\Big] \tag{7.2}$$

which has the form of (4.2) with $f_0(x)$ being again the logistic map (2.10) and

$$f_1(x) = -(2x-1)^2 \ln(2x-1)^2 \tag{7.3}$$

Obviously the expansion (7.2) cannot be convergent when $x \approx 1/2$, i.e., in the vicinity of the maximum point of $f(x)$ and consequently one cannot hope that a direct application of the perturbation theory based on it can lead to meaningful results. The difficulty, however, can be overcome by exploiting the results of Section 3. To demonstrate how this works we consider Eq. (4.5) with $k = 1$ for the first correction to the invariant measure. The function $F_1(x)$ occurring there can be determined by using (7.3) and (A.3):

$$F_1(x) = \frac{1}{\pi}\left(\frac{1-x}{x}\right)^{1/2} \ln(1-x) \tag{7.4}$$

Since $f_0(x)$ maps the region around $x = 1/2$ into the neighborhood of the point $x = 1$, Eq. (4.5) with $k = 1$ and with the functions (7.3) and (7.4) possesses the asymptotic solution

$$\mu_1(x) = \frac{1}{\pi}(1-x)^{1/2}\ln(1-x) - \mu_1'(1/2)(1-x)^{1/2}, \qquad x \to 1 \tag{7.5}$$

It can then be easily seen that

$$\mu_1(x) = -\frac{1}{\pi}\sqrt{x}\ln x + \left[\mu_1'(1/2) - \frac{2}{\pi} + \frac{4}{\pi}\ln 2\right]\sqrt{x} \tag{7.6}$$

satisfies the same equation asymptotically near $x = 0$. Taking into account (4.23) we obtain

$$\mu(x) = \frac{2}{\pi}\sqrt{x}\left\{1 + \epsilon\left[\frac{\pi}{2}\mu_1'(1/2) - 1 + 2\ln 2\right] - \frac{\epsilon}{2}\ln x + \mathscr{O}(\epsilon^2)\right\}, \qquad x \to 0 \tag{7.7}$$

and

$$\mu(x) = 1 - \frac{2}{\pi}(1-x)^{1/2}\Big[1 + \epsilon\frac{\pi}{2}\,\mu_1'(1/2)$$

$$- \frac{\epsilon}{2}\ln(1-x) + \mathscr{O}(\epsilon^2)\Big], \qquad x \to 1 \quad (7.8)$$

while according to Eqs. (3.2), (3.4), and (3.7) the correct asymptotic behavior of the invariant measure for the map (7.1) to leading order in ϵ is

$$\mu(x) = \left\{\frac{2}{\pi} + \epsilon\left[\mu_1'(1/2) - \frac{2}{\pi} + \frac{4}{\pi}\ln 2\right]\right\} x^{(1-\epsilon)/2}, \qquad x \to 0 \quad (7.9)$$

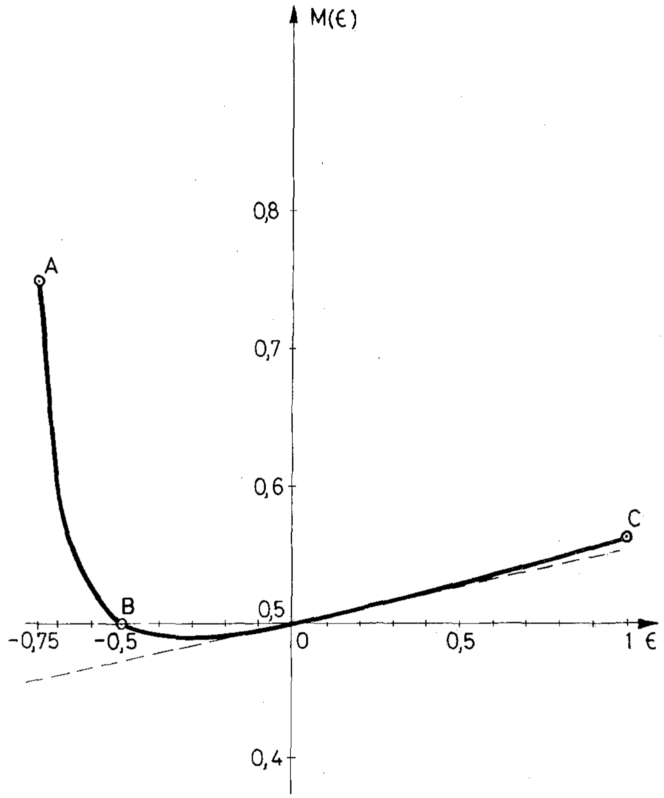

Fig. 4. The asymmetry number M for maps $f(\epsilon, x) = 1 - |2x - 1|^{2(1+\epsilon)}$ as a function of ϵ. The solid curve corresponds to iteratively computed values, while the theoretical line (7.11) is dotted. The points A, B, and C represent the map (7.12), the piecewise linear map, and the purely quartic map, respectively.

and

$$\mu(x) = 1 - \left[\frac{2}{\pi} + \epsilon\mu_1'(1/2) \right](1-x)^{(1-\epsilon)/2}, \qquad x \to 1 \tag{7.10}$$

This means that the logarithms appearing in the perturbation calculation near $x = 0$ and $x = 1$ should be exponentiated. Since we want to have a smooth function for the measure, the fitting of the corrected asymptotic solutions, valid for $x \to 0$, $x \to 1$, to the solution in the inner part of the interval $[0, 1]$ calculated from (4.5) necessitates some modification of the latter one. Concerning the first-order calculation discussed above, this modification would contain terms of $\mathcal{O}(\epsilon^2)$ and smaller, and therefore with the accuracy we want to calculate is negligible.

As an example we consider below the asymmetry number for which according to (5.7) the value of the invariant measure at $x = 1/2$ is needed. Solving numerically the equation for the first-order correction we find

$$M(\epsilon) = 1/2 + \epsilon \cdot 0.0542 \cdots + \mathcal{O}(\epsilon^2) \tag{7.11}$$

We computed $M(\epsilon)$ by means of the iteration of (6.2), too, and drew the results on Fig. 4. Point A corresponds to the map

$$f(-3/4, x) = 1 - (|2x - 1|)^{1/2} \tag{7.12}$$

the probability distribution of which can be given explicitly:

$$P(-3/4, x) = 2(1 - x) \tag{7.13}$$

This can be checked by direct substitution into Eq. (2.4). The LCE for the map (7.12) is equal to $1/2$.

Finally, by substituting (7.11) into (5.9) the inequality

$$\lambda \leqslant \ln 2 - 0.00588 \ldots \cdot \epsilon^2 + \mathcal{O}(\epsilon^3) \tag{7.14}$$

is obtained for the Lyapunov characteristic exponent.

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APPENDIX

Assuming the expansions (4.2) and (4.3) for the map $f(\epsilon, x)$ and the invariant measure $\mu(\epsilon, x)$, respectively, we calculate the explicit form of Eq. (4.5) for the kth-order correction $\mu_k(x)$. Suppose that each correction $\mu_k(x)$ is analytic and consider the left-hand side of Eq. (4.1)

$$\begin{aligned}\mu(\epsilon, f(\epsilon, x)) &= \mu_0(f_0(x) + \epsilon f_1(x) + \cdots) \\ &\quad + \epsilon\mu_1(f_0(x) + \epsilon f_1(x) + \cdots) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{\infty} \epsilon^j \mu_j^{[n]}(f_0(x)) \left[\sum_{i=1}^{\infty} \epsilon^i f_i(x) \right]^n \\ &\equiv \sum_{n=0}^{\infty} A_n \end{aligned} \tag{A.1}$$

where the notation $\mu_j^{[n]}(x) = d^n\mu_j(x)/dx^n$ has been used. Writing

$$A_n = \frac{1}{n!} \sum_{i_1, \ldots, i_n = 1}^{\infty} \sum_{j=0}^{\infty} \epsilon^{i_1 + \cdots + i_n + j} f_{i_1}(x) \cdots f_{i_n}(x) \mu_j^{[n]}(f_0(x))$$

and collecting in $A_n - s$, $n \leqslant k$, the terms of kth order in ϵ, after some rearrangements we obtain

$$\mu(\epsilon, f(\epsilon, x)) = \sum_{k=0}^{\infty} \epsilon^k T_k(x)$$

where

$$T_k(x) \equiv \mu_k(f_0(x)) - F_k(f_0(x))(1 - \delta_{k0}) \tag{A.2}$$

The function $F_k(x)$ is given by

$$F_k(x) = -\sum_{n,l=1}^{k} \mu_{k-l}^{[n]}(x) \sum_{\{a_i\}}^{(n,l,k)} \prod_{i=1}^{k} \frac{1}{a_i!} f_i^{a_i}(f_0^{-1}(x)) \tag{A.3}$$

where the symbol $\sum_{\{a_i\}}^{(n,l,k)}$ indicates the following summation:

$$\sum_{\{a_i\}}^{(n,l,k)} = \sum_{a_1=0}^{k} \sum_{a_2=0}^{k-1} \cdots \sum_{a_k=0}^{1} \tag{A.4a}$$

with the constraints

$$\sum_{i=1}^{k} a_i = n, \qquad \sum_{i=1}^{k} i a_i = l \tag{A.4b}$$

We have already taken into account the second constraint to the extent that we have put the possible upper limits of the summations over $a_i - s$ allowed by this constraint. Note that there is no ambiguity in (A.3) which branch of the inverse of $f_0(x)$ has to be used, since the $f_i(x) - s$ are all symmetric.

Expanding the right-hand side of Eq. (4.1) one has

$$\mu(\epsilon, x) - \mu(\epsilon, 1-x) = \sum_{k=0}^{\infty} \epsilon^k \big(\mu_k(x) - \mu_k(1-x) \big) \tag{A.5}$$

The comparison of (A.2) and (A.5) yields Eq. (4.5) with $F_k(x)$ given by (A.3).

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